UNSTEADY MOTION OF ROTATING RING OF VISCOUS INCOMPRESSIBLE LIQUID WITH FREE BOUNDARY

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We examine the plane problem of rotationally symmetric motion of a rotating ring of viscous incompressible liquid with free boundary. The theorem for the existence and uniqueness of the problem solution is obtained. The qualitative properties of the solution and its asymptotic behavior as $t \to \infty$ are studied.

1. Formulation of the Problem. Let a viscous liquid fill at the initial moment the ring $R_{20} < r < R_{10}$ and have a given distribution, having rotational symmetry, of the radial and angular velocities.

We are required to study the inertial motion of the ring. The initial conditions have rotational symmetry, therefore it is natural to seek the solution of the problem in this class. It will be shown later that the problem is uniquely solvable in this class.

Whether the problem has a solution not having axial symmetry is not known. As the mathematical model of the motion we use the system of Navier-Stokes equations, which is to be solved in the region $t > 0$, $R_2(T) < r < R_1(t)$. Here $r = R_{1,2}(t)$ are respectively the outer and inner boundaries of the ring, which are initially unknown. The equation of continuity in polar coordinates $d(rv_r)/dr = 0$ is easily integrated and yields

$$v_r = r^{-1}\Phi(t) \tag{1.1}$$

Therefore the Navier-Stokes equations in polar coordinates can be written as

$$\frac{1}{r}\frac{\partial\Phi}{\partial t} - \frac{\Phi^2}{r^3} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} \tag{1.2}$$

$$\frac{\partial v}{\partial t} + \frac{\Phi}{r}\frac{\partial v}{\partial r} + \frac{\Phi v}{r^2} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{1.3}$$

Here $v \equiv v_0$ and $\rho = 1$, which does not impair the generality. We obtain the following edge conditions from vanishing of the stress vector at the free boundary

$$\begin{aligned} \tau_{rr} &= -p - \frac{2\nu\Phi}{r^2} = 0 \\ \tau_{r\theta} &= \frac{\partial v}{\partial r} - \frac{v}{r} = 0 \qquad \text{for} \quad r = R_{1,2}(t) \end{aligned} \tag{1.4}$$

From (1.2) with the aid of (1.4) we find

$$\int\limits_{R_2(t)}^{R_1(t)} \frac{v^2}{r}\,dr - \left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)\left(2\nu\Phi - \frac{\Phi^2}{2}\right) - \frac{d\Phi}{dt}\ln\frac{R_1}{R_2} = 0 \tag{1.5}$$

We add the initial condition to (1.5)

$$\Phi(0) = \Phi_0. \tag{1.6}$$

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The kinematic condition at the free boundary

$$\frac{d\,[R_{1,2}(t)]}{dt} = \frac{\Phi(t)}{R_{1,2}(t)} \tag{1.7}$$

yields the area integral

$$R_1^2 - R_2^2 = R_{10}^2 - R_{20}^2, \qquad R_{i0} = R_i(0) \qquad (i = 1,2)$$

The initial and edge conditions for (1.3) have the form

$$v(r,\ 0) = v_0(r) \tag{1.8}$$

$$\frac{\partial v}{\partial r} - \frac{v}{r} = 0, \quad t \geqslant 0, \quad r = R_{1,2}(t) \tag{1.9}$$

We note that the moment of momentum conservation law is satisfied for (1.3)

$$\int_{R_2(t)}^{R_1(t)} r^2 v(r,\ t)\, dr = \int_{R_{20}}^{R_{10}} r^2 v_0(r)\, dr \tag{1.10}$$

In the obtained system of equations we convert to new independent variables and unknown functions using the equations

$$\eta = \frac{r^2 - R_2^2(t)}{R_{20}^2}, \qquad t = \frac{R_{20}^2}{\nu}\tau$$
$$v = \frac{\nu}{R_{20}^2} r\omega, \quad \xi = \frac{R_2^2(t)}{R_{20}^2}, \quad \psi = \frac{\Phi}{\nu} \tag{1.11}$$

Moreover, we introduce the notation

$$a = R_{10}^2 / R_{20}^2 - 1$$

Then the problem (1.3), (1.5)-(1.9) takes the form

$$\frac{\partial \omega}{\partial \tau} + \frac{2\psi}{\xi + \eta}\omega = 4(\xi + \eta)\frac{\partial^2 \omega}{\partial \eta^2} + 8\frac{\partial \omega}{\partial \eta} \tag{1.12}$$

$$\omega|_{\tau=0} = \omega_0(\eta) \qquad (0 \leqslant \eta \leqslant a) \tag{1.13}$$

$$\partial\omega / \partial\eta = 0, \ \tau \geqslant 0, \ \eta = 0, \ a \tag{1.14}$$

$$\frac{d\psi}{d\tau} = \frac{a\psi(\psi - 4)}{\xi(\xi + a)\ln(1 + a/\xi)} + \frac{1}{\ln(1 + a/\xi)}\int_0^a \omega^2\, d\eta \tag{1.15}$$

$$d\xi / d\tau = 2\psi, \ \psi(0) = \psi_0, \ \ \xi(0) = 1 \tag{1.16}$$

We note that as a result of the replacement we can convert from a boundary problem with unknown boundary to a problem in the fixed region $\{0 \le \eta \le a, \tau \ge 0\}$.

2. Radial Motion of the Ring. If $\omega \equiv 0$, the problem (1.12)-(1.16) simplifies considerably and reduces to the Cauchy problem (1.15), (1.16).

Taking ξ as the independent variable in (1.15) and using (1.16), we have

$$\frac{d\psi}{d\tau} = \frac{d\psi}{d\xi}\frac{d\xi}{d\tau} = 2\psi\frac{d\psi}{d\xi}, \quad \frac{d\psi}{d\xi} = \frac{a(\psi - 4)}{2\xi(\xi + a)\ln(1 + a/\xi)} \tag{2.1}$$

$$\psi|_{\xi=1} = \psi_0 \tag{2.2}$$

It will be shown later that the sign of $d\xi/d\tau$ for all $\tau \ge 0$ coincides with the sign of

$$\left.\frac{d\xi}{d\tau}\right|_{\tau=0} = 2\psi_0$$

The case $\psi_0 \ge 0$ describes an expanding ring, and $\psi_0 < 0$ describes a contracting ring.

Integrating (2.1) with the condition (2.2), we obtain

$$\psi = 4 + \frac{(\psi_0 - 4)\sqrt{\ln(1+a)}}{\sqrt{\ln(1+a/\xi)}} \tag{2.3}$$

Let $\psi_0 < 0$, then for

$$\xi = \xi_0^* = \frac{a}{(1+a)^{\varkappa} - 1}, \qquad \varkappa = \left(\frac{4-\psi_0}{2}\right)^2$$

the function ψ vanishes and $0 < \xi_0^* < 1$ for any $\psi_0 < 0$. This means that the ring contracts to a critical radius. As a result of (1.16)

$$\frac{1}{2}\int_1^{\xi(t)} \frac{d\xi}{\psi(\xi)} = t \tag{2.4}$$

It follows from (2.3) that the time for the ring to contract to the critical radius is infinite, and the velocity approaches zero exponentially.

Let $\psi_0 > 0$, which corresponds to expansion of the ring. Three cases are possible:

1) If $\psi_0 > 4$, then $\psi = O(\sqrt{\xi})$. Then $R_2(t) \sim t$ and the ring expansion rate is asymptotically constant;

2) if $\psi_0 = 4$, then $\psi \equiv 4$. In this case $R_2(t) \sim \sqrt{t}$ and the rate expansion $R_2' \sim 1/\sqrt{t}$;

3) if $\psi_0 < 4$ then the ring expands after an infinite time to the critical radius and the rate of expansion approaches zero exponentially;

3. A priori Estimates. Let $\xi(\tau)$, $\psi(\tau)$ be continuous functions and $\xi(\tau) > 0$, then the solution ω of the problem (1.12)-(1.14) satisfies the following energy estimate

$$\int_0^a \omega^2 d\eta \leqslant \frac{ca}{\xi(\xi + a)} \tag{3.1}$$

For proof we multiply (1.12) by $(\xi + \eta)\omega$. We obtain

$$\frac{1}{2}\frac{\partial}{\partial\tau}[(\xi+\eta)\,\omega^2] + \frac{1}{2}\xi'\omega^2 = 4\frac{\partial}{\partial\eta}[(\xi+\eta)^2\omega\omega_\eta] - 4(\xi+\eta)^2\omega_\eta^2$$

Here $\xi' \equiv d\xi/d\tau = 2\psi$ in accordance with (1.16). Let us integrate this equality with respect to η from 0 to a. Using the edge conditions (1.14) and the nonnegativity of $(\xi + \eta)^2\omega_\eta^2$, we obtain

$$\frac{d}{d\tau}\int_0^a (\xi+\eta)\,\omega^2 d\eta + \xi'\int_0^a \omega^2 d\eta \leqslant 0$$

But

$$\int_0^a \omega^2 d\eta \geqslant \frac{1}{\xi + a}\int_0^a (\xi+\eta)\,\omega^2 d\eta$$

Therefore

$$\frac{d}{d\tau}\int_0^a (\xi+\eta)\,\omega^2 d\eta + \frac{\xi'}{\xi + a}\int_0^a (\xi+\eta)\,\omega^2 d\eta \leqslant 0$$

We introduce the notation

$$\int_0^a (\xi+\eta)\,\omega^2 d\eta \equiv J$$

Then by virtue of the nonnegativity of J we obtain from the last inequality

$$J \leqslant \frac{ca}{\xi + a}\left(c = \frac{1}{a}\int_0^a (1+\eta)\,\omega_0^2(\eta)\,d\eta\right) \tag{3.2}$$

Since

$$J \geqslant \xi \int_0^a \omega^2 d\eta$$

the required estimate is obtained. We note that for the solution ω of the problems (1.12)-(1.14) the maximum principle in the following form is valid [2]. For any continuous $\xi(\tau)$, $\psi(\tau)$, $\xi(\tau) > 0$, $|\psi| < \infty$ the function $u = (\xi + \eta)\omega$ reaches for an extremum either for $\tau \geq 0$, $\eta = 0, a$ or $\tau = 0$, $0 \leq \eta \leq a$.

Now let us establish a priori estimates for ξ and ψ. Let us examine the case $\psi_0 < 0$. Taking ξ as the independent variable in (1.15), we obtain

$$2\psi \frac{d\psi}{d\xi} = \frac{a\psi(\psi - 4)}{\xi(\xi + a)\ln(1 + a/\xi)} + \frac{1}{\ln(1 + a/\xi)} \int_0^a \omega^2 d\eta \tag{3.3}$$

In this case (2.1) is the majorizing equation for (3.3). We denote by ψ_1 the solution of (2.1) satisfying the same initial condition as does ψ. On the basis of a Chaplygin type differential inequality we obtain $\psi_1 < \psi$. The derivatives

$$d\xi / d\tau = 2\psi, \qquad d\xi_1 / d\tau = 2\psi$$

therefore

$$0 < \xi_1(\tau) < \xi(\tau)$$

As the minorant we can take $\xi = 1$, since $d\psi/d\xi$ is bounded in the interval $|\xi_0^*, 1|$.

Hence we conclude that for $\psi_0 < 0$ and any continuous $\omega_0(\eta)$ there exists ξ^*, $\xi_0^* < \xi^* \leq 1$ such that $\psi(\xi^*) = 0$. We can obtain a more exact upper estimate for ξ^* if we use the minorizing equation

$$2\varphi \frac{d\varphi}{d\xi} = \frac{a\varphi(\varphi - 4)}{\xi(\xi + a)\ln(1 + a/\xi)} + \frac{ac}{\xi(\xi + a)\ln(1 + a/\xi)} \tag{3.4}$$

where C is defined by (3.2). This estimate is not written out because of its complexity. It can be shown that for $\psi_0 < 0$ the rotating ring contracts to the critical radius after a finite time.

In fact, let $\xi \to \xi^*$, then the first term in the right side of (3.3) approaches zero while the second term approaches a finite, nonzero value. If it were equal to zero, then as a result of the maximum principle for the function $u = (\xi + \eta)\omega$ we find that $\omega \equiv 0$.

Therefore $\psi = O(\sqrt{\xi^* - \xi})$ and the integral

$$\int_1^{\xi(t)} \frac{d\xi}{\psi(\xi)} = t$$

converges as $\xi \to \xi^*$. Hence follows the statement to be proved.

In the case $\psi_0 \geq 0$ the inequality (3.1) makes it possible to write the majorizing equation for (3.3); (2.1) will be the minorizing equation. Thereby we obtain the a priori estimate $|\psi| \leq K(T)$ for $(0, T)$ with any $T < \infty$. After the estimates for (3.1) are obtained, and also the estimates for $\psi(\tau)$, we can turn to proof of the theorem on existence and uniqueness.

4. Theorem on Existence and Uniqueness. For any continuous $\omega_0(\eta)$ and for any ψ_0 there exists a unique solution of the problems (1.12)-(1.16) for all $t \geq 0$.

We shall present an abbreviated proof. We introduce the space $C = C[0, \tau_1]$, whose elements are continuous on $[0, \tau_1]$ vector functions $\lambda\{\xi(\tau), \psi(\tau)\}$ with the norm

$$\|\lambda\| = \max_\tau \{\max|\xi|, \ |\psi|\}$$

We specify $\lambda_1\{\xi(\tau), \psi_1(\tau)\}$, continuous on $[0, \tau_1]$, and we substitute ξ_1 and ψ_1 in place of ξ, ψ in (1.12). Then the function ω is uniquely defined as the solution of the second boundary problem (1.12)-(1.14) ([2], Chapter 5) and we have the estimate

$$\int_0^a \omega^2 d\eta \leqslant K(\tau_1) \quad (\xi \geqslant \delta > 0)$$

From the value found for ω we determine ξ, ψ as the solution of the Cauchy problems (1.15), (1.16). Thereby we determine the operator T which makes the pair of functions $\{\xi_1, \psi_1\}$ correspond to the pair of of functions $\{\xi, \psi\}$. By virtue of the a priori estimates obtained for ξ, ψ and also (3.1), this operator maps the sphere $||\lambda - \lambda_0|| < k$ of the space $C[O, \tau_1]$ onto itself. Here $\lambda_0 = \{1, \psi_0\}$. Moreover, T is a contracting operator; this is easily achieved as a result of the smallness of τ_1. It is clear that the existence of a stationary point of the operator T proves the theorem on the existence and uniqueness of the problem (1.12)-(1.18) for sufficiently small τ_1. The existence of a uniform a priori estimate for

$$\int_0^a \omega^2 d\eta, \ \xi(\tau), \ \psi(\tau)$$

makes it possible to obtain by repeated application of these arguments the existence and uniqueness theorem for the interval (0, T) with an $T < \infty$.

5. Qualitative Description of the Motion. We have obtained certain qualitative results: the ring always contracts to the critical radius after a finite time except for the case $\omega \equiv 0$. In this case the contraction time is infinite. After the ring contracts to the critical radius it begins to expand. This case is more complex.

Equation (3.4) is majorizing for (3.3) with $\xi \geq 1$. Let us examine for it the Cauchy problem with the initial condition $\varphi|_{\xi=1} = \psi_0$. This problem is solved explicitly

$$\int_{\psi_0}^{\varphi} \frac{2\chi d\chi}{\chi^2 - 4\chi + c} = \ln \frac{\ln(1 + a/\xi)}{\ln(1 + a)}$$

In view of their complexity the final equations are not written out. The following three cases are possible.

First Case $c > 4$. Then $\varphi = 0(\sqrt{\xi})$ as $\xi \to \infty$. If $\psi_0 > 4$ the minorant (2.3) behaves the same as $\xi \to \infty$, therefore the solution as $\xi \to \infty$ in this case is also $\psi = O(\sqrt{\xi})$. Further

$$\frac{d[R_2(t)]}{dt} = \frac{\psi}{R_2(t)}, \quad \xi = \frac{R_2^2(t)}{R_{20}^2}$$

Consequently $R_2(t) = O(t)$ as $t \to \infty$.

Second Case $C = 4$. Then:

a) if $\psi_0 > 2$, then as before

$$\varphi = O(\sqrt{\xi}) \quad \text{for} \quad \xi \to \infty$$

b) if $\psi_0 = 2$, then we find directly from (5.1)

$$\varphi \equiv 2 \quad \text{or} \quad \psi(\xi) \leqslant 2$$

c) if $\psi_0 < 2$, then $\varphi \to 2$ as $\xi \to \infty$.

Third Case $c < 4$. We introduce the notation $b = \sqrt{4 - C}$.

a) if $\psi_0 > 2 + b$, then $\varphi = O(\sqrt{\xi})$ as $\xi \to \infty$

b) if $2 - b < \psi_0 < 2 + b$, then $\varphi(\xi)$ is bounded

$$\varphi(\xi) \to 2 - b > \quad \text{for} \quad \xi \to \infty$$

c) if $\psi_0 < 2 - b$, then $\varphi(\xi) \to 2 - b$, however, while in the preceding case $\varphi(\xi)$ approached $2 - b$ from above, in this subcase it approaches from below.

For $\psi_0 < 4$ the minorant vanishes for some finite ξ.

We shall show that the solution ψ of (3.3) does not vanish for any finite ξ.

Let us examine (3.3). Let $\xi_1 > 1$ be the first value for which $\psi = 0$. As $\xi \to \xi_1$ from the left we have $\psi \to +0$, and the integral

$$\int_0^a \omega^2 d\eta \to \text{const} > 0$$

(The positivity of this limit follows from the maximum principle (see Section 2), since otherwise $\omega \equiv 0$). On the other hand, the limit of the left side of (3.3) obviously cannot be positive as $\xi \to \xi_1 - 0$, which proves the statement made.

We see from the preceding analysis that there are two qualitatively different expansion regimes. In the first regime $R_2(t) = O(t)$. This regime corresponds to expansion either for $\psi_0 \geq 4$ and any c, where ψ_0 plays the role of the radial Reynolds number; for $c > 4$ and any $\psi_0 \geq 0$, where c plays the role of the angular Reynolds number, or for $c = 4$ and $\psi_0 < 2$. Such asymptotic behavior is characteristic for the potential motion of a rotating ideal liquid ring [3]. In the second regime $R_2(t) = O(\sqrt{t})$. This occurs either for $\psi_0 < 4$, $c < 4$ or for $\psi_0 < 2$, $c = 4$.

In fact, in this case the solution of the majorizing equation is bounded for all t, therefore

$$\frac{d[R_2(t)]}{dt} \leqslant \frac{K}{R_2(t)}, \quad R_2(t) = O(\sqrt{t})$$

We note the following interesting fact. Let $R_{20} = 0$, then the governing equations admit the stationary solution

$$v = \Omega r, \quad \Phi = 0, \quad p = -\tfrac{1}{2}\Omega^2(R_1^2 - r^2)$$

which corresponds to rotation of the ring as a solid body, but it is sufficient to take $R_{20} \neq 0$ even arbitrarily small for the picture to change markedly: There are in general no stationary solutions other than

$$v = 0, \quad \Phi = 0, \quad p = 0$$

6. Motion of Ideal Liquid Ring. Ovsyannikov [3] examined the potential motion of a rotating ideal liquid ring. In the following we examine the more general case of vortical motion. For $\nu \equiv 0$ the basic equations simplify, and after replacing the independent variables and the unknown functions using the equations

$$\xi = \frac{R_2^2(t)}{R_{20}^2}, \quad \eta = \frac{r^2 - R_2^2(t)}{R_{20}^2}, \quad U = \frac{R_{20}}{\Phi_0} v, \quad W = \frac{\Phi}{\Phi_0}$$

they can be written in the form

$$\frac{\partial u}{\partial \xi} + \frac{U}{\xi + \eta} = 0, \qquad U(\eta, 1) = U_0(\eta) \tag{6.1}$$

$$\frac{d}{d\xi}\left[W^2 \ln \frac{\xi + a}{\xi}\right] = \int_0^a U^2 d\eta, \quad W|_{\xi=1} = W_0 \quad \left(a = \frac{R_{10}^2}{R_{20}^2} - 1\right) \tag{6.2}$$

Problems (6.1) and (6.2) are solved sequentially and yield

$$U = \frac{u_0(\eta)(1+\eta)}{\xi + \eta}, \quad W^2 = \frac{1}{\ln(1 + a/\xi)}\left[C_0 + \int_1^\xi \left(\int_0^a U^2 d\eta\right) d\xi\right]$$

$$C_0 = W_0^2 \ln(1 + a) - \left[\int_1^\xi \left(\int_0^a U^2 d\eta\right) d\xi\right]_{\xi=1}$$

It is clear that for any continuous on $[0, a]$ functions $U_0(\eta)$ and $\xi(T) < 1$ there exists an $0 < \xi^* < 1$ such that $w(\xi^*) = 0$.

As $\xi \to \xi^*$ we have

$$w(\xi) = K\sqrt{\xi - \xi^*}[1 + O(\xi - \xi^*)] \quad (k \neq 0)$$

Considering that

$$t = \int_1^\xi \frac{d\xi}{W(\xi)}$$

we find that the ring contracts to the critical radius after a finite time. However, if $W_0 > 0$, then $W(\xi) = O(\sqrt{\xi})$ as $\xi \to \infty$. This means that $R_2(t) = O(t)$ as $t \to \infty$, which is analogous to the asymptotic behavior of the potential motion [3].

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